

SPECTRA ASSOCIATED TO SYMMETRIC MONOIDAL BICATEGORIES

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ABSTRACT. We show how to construct a Γ -bicategory from a symmetric monoidal bicategory, and use that to show that the classifying space is an infinite loop space upon group completion. We also show a way to relate this construction to the classic Γ -category construction for a bipermutative category. As an example, we use this machinery to construct a delooping of the \mathcal{K} -theory of a bimonoidal category as defined in [BDR].

1. INTRODUCTION

Symmetric monoidal bicategories appear in many contexts in mathematics. Some examples include *Bimod*, the bicategory of rings, bimodules and homomorphisms of bimodules, with tensor product as the monoidal structure; and *nCob*, the bicategory of closed n -manifolds, cobordisms and diffeomorphisms between cobordisms, with disjoint union as the monoidal structure. The definition of symmetric monoidal bicategory is cumbersome. The most concise definition can be found in [SP]. Shulman [Shu] shows how to obtain symmetric monoidal bicategories from symmetric monoidal double categories, which are easier to understand.

In view of the importance of the construction of spectra from symmetric monoidal categories [May1, Seg], we can ask then if there is a similar construction for symmetric monoidal bicategories. More specifically, we would like to know if the group completion of the classifying space of a symmetric monoidal bicategory is an infinite loop space. In this paper we show that this is the case for *strict* symmetric monoidal bicategories. To do this, we use the theory of Segal's Γ -spaces [Seg]. This procedure gives a functor from symmetric monoidal bicategories to spectra that we call \mathbb{A} .

Given a bimonoidal category \mathcal{R} , we construct the bicategory of \mathcal{R} -modules, $\mathcal{M}(\mathcal{R})$. It turns out that the \mathcal{K} -theory of \mathcal{R} , as defined in [BDR] is the group completion of the classifying space of $\mathcal{M}(\mathcal{R})$. We prove that $\mathcal{M}(\mathcal{R})$ is a strict symmetric monoidal bicategory, and hence, $\mathcal{K}(\mathcal{R}) = \mathbb{A}_0(\mathcal{M}(\mathcal{R}))$ is an infinite loop space. We compare this result with those of [BDRR], where the authors obtain an equivalence of spaces

$$(1.1) \quad \mathcal{K}(\mathcal{R}) \xrightarrow{\sim} K(\mathbb{E}(\mathcal{R})),$$

where the right-hand side is the zeroth space of the algebraic K -theory spectrum of the ring spectrum $\mathbb{E}(\mathcal{R})$. We prove that this equivalence is an equivalence of infinite loop spaces.

1.1. Organization. Section 2 contains the necessary background information on bicategories and Segal's Γ -spaces. We state the main results in Section 3. Section 4.2 is dedicated to the application of the main results to the K -theory of bimonoidal categories. The remaining sections contain the main proofs and constructions. Appendix A gives an account of the construction of the classifying space of a bicategory.

1.2. Notation. We denote the functor from the category of (small) permutative categories to the category of spectra by \mathbb{E} . Let R be a ring spectrum. Let $\mathcal{M}(R)$ be the category of finitely generated free modules over R . Then the K -theory spectrum of R is defined as $\mathbb{K}(R) = \mathbb{E}(\mathcal{M}(R))$. The K -theory space is the zeroth space of this spectrum and it is denoted by $K(R)$.

1.3. Warning! Both monoidal bicategories and bimonoidal categories appear in this paper. We hope that the similarity of these terms leads to no confusion.

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2. BACKGROUND

2.1. Bicategories. In this paper we will be working with bicategories. For definitions and proofs of the basic theorems, we refer the reader to the earlier papers on bicategories [Bén, Str] and some more recent accounts [Lei, SP].

Throughout the document we will assume categories and bicategories are enriched over simplicial sets without explicitly saying it. A bicategory is enriched over simplicial sets if all the categories of morphisms are enriched. Whenever we work with a non-enriched category we will indicate it explicitly.

Given a bicategory \mathcal{C} , we will denote the vertical composition by \circ , the horizontal composition by $*$. For an object A in \mathcal{C} , I_A denotes the identity 1-morphism. The associativity isomorphism is denoted by α , while the right and left identity isomorphisms are denoted by r and l , respectively.

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a pseudofunctor. We denote the functoriality 2-isomorphisms by

$$F_{f,g}^2 : F(g) * F(f) \rightarrow F(g * f) \quad \text{and} \quad F_A^0 : I_{FA} \rightarrow F(I_A).$$

These are subject to coherence axioms as stated in [Lei, Section 1.1].

Given a transformation $\eta : F \rightarrow G$ between pseudofunctors we denote the 1-morphism by $\eta_A : FA \rightarrow GA$ and the naturality 2-isomorphism by $\eta_f^2 : Gf * \eta_A \rightarrow \eta_B * Ff$.

We will make extensive use of pasting diagrams for bicategories. A pasting diagram is a polygonal arrangement on the plane, where the vertices correspond to objects, the directed edges correspond to 1-morphisms and the faces are usually filled with double arrows corresponding to 2-morphisms. For example, the diagram

indicates that σ is a 2-morphism from $g * f$ to h .

We can combine pasting diagrams to depict certain compositions of 2-morphisms. For example the diagram

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represents the 2-morphism given by the composition

$$k * f \xRightarrow{k * \varphi} k * (h * g) \xRightarrow{\alpha} (k * h) * g \xRightarrow{\psi * g} l * g.$$

We note here that the 2-morphisms are not actually composable; we need to use the associativity isomorphism. In general, for larger diagrams, the source and target of the 2-morphisms we are composing may differ by their bracketing. By the Coherence Theorem of Bicategories [MLP] we know that there is a unique canonical associativity isomorphism between two bracketings, so we use this isomorphisms to connect the source and target of the 2-morphisms we are composing and hence make sense of the diagram.

Once we specify a bracketing of the outside 1-morphisms, the diagram has a unique meaning, no matter what order we use to compose the 2-morphisms. We refer the reader to [KS].

When we say “pasting diagram A is equal to pasting diagram B ” we mean that with a given bracketing of the outside 1-morphisms, the given 2-morphisms that they both define are equal. Note that if this is true for a given bracketing, it is true for all bracketings.

2.2. Γ -spaces. Segal’s Γ -spaces give an infinite loop space machine. In [Seg], it is shown that a symmetric monoidal category \mathcal{M} gives rise to a Γ -space, and hence a cohomology theory. We will recall briefly what a Γ -space is, since the definition will play a central role in the paper.

Let Fin_* denote the (skeletal) category of finite pointed sets and pointed maps. The skeletal version has as objects the sets $\underline{n} = \{0, 1, \dots, n\}$, for $n \geq 0$. Here 0 is the basepoint.

For $1 \leq k \leq n$, we define $i_k : \underline{n} \rightarrow \underline{1}$ as:

$$i_k(j) = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k. \end{cases}$$

Definition 2.1. A Γ -space X is a functor $X : Fin_* \rightarrow Top$. We say X is *special* if the map

$$P_n : X(\underline{n}) \rightarrow X(\underline{1})^{\times n},$$

obtained by assembling the maps i_k , is a weak equivalence for all $n \geq 0$.

The conditions in the definition above roughly imply that the space $X(\underline{1})$ has a multiplication that is associative and commutative up to coherent higher homotopies. The precise statement is given by the following theorem:

Theorem 2.2. [Seg, Prop. 1.4] *Let X be a special Γ -space. Then $X(\underline{1})$ is an H -space and its group completion, $\Omega BX(\underline{1})$ is an infinite loop space.*

Segal and May [May1] show how to construct a Γ -category from a symmetric monoidal category, thus getting an infinite delooping of the classifying space of a symmetric monoidal category. As we mention in the introduction, given a permutative category \mathcal{M} , we denote the associated spectrum by $\mathbb{E}(\mathcal{M})$. We will follow a similar approach in the context of bicategories.

3. STATEMENT OF RESULTS

In broad terms, a symmetric monoidal bicategory is a bicategory with a product pseudo-functor that is associative, unitary, and commutative up to coherent natural equivalences.

The precise definition is quite involved, as one would imagine. For a precise definition and a historical account of the theory of symmetric monoidal bicategories we refer the reader to [SP]. Shulman [Shu] provides a way of constructing examples of symmetric monoidal bicategories.

In this paper we will be working with a strict version of symmetric monoidal bicategories, so that is what we will define here.

Definition 3.1. A *strict symmetric monoidal bicategory* $(\mathcal{M}, \boxplus, 1, \beta)$ consists of the following data:

- a bicategory \mathcal{M} ;
- a pseudofunctor of bicategories $\boxplus : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$;
- an object 1 in \mathcal{M} called the *unit*;
- a transformation $\beta : \boxplus \rightarrow \boxplus \circ \tau$, where τ denotes the twist pseudofunctor $\mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M} \times \mathcal{M}$.

The monoidal product \boxplus is required to be strictly associative, and 1 is a strict unit. The following diagrams must commute:

$$\begin{array}{ccc}
 A \boxplus B \boxplus C & \xrightarrow{\beta \boxplus I} & B \boxplus A \boxplus C \\
 & \searrow \beta & \downarrow I \boxplus \beta \\
 & & B \boxplus C \boxplus A
 \end{array}
 \qquad
 \begin{array}{ccc}
 A \boxplus B \boxplus C & \xrightarrow{I \boxplus \beta} & A \boxplus C \boxplus C \\
 & \searrow \beta & \downarrow \beta \boxplus I \\
 & & C \boxplus A \boxplus B
 \end{array}$$

$$\begin{array}{ccc}
 A \boxplus B & \xlongequal{\quad} & A \boxplus B \\
 & \searrow \beta & \nearrow \beta \\
 & B \boxplus A &
 \end{array}$$

Remark 3.2. The strict version defined above is just one of the many ways in which one can strictify the notion of symmetric monoidal bicategory. In fact, there are different levels of strictness one could consider. We choose this level because it is convenient to work with and it covers the applications we have in mind. It is easy to check that this definition is a special case of the general definition of symmetric monoidal bicategory. On the other hand, it is not yet known whether or not a general symmetric monoidal category can be strictified.

Indeed, it is known that any monoidal bicategory is equivalent to a Gray monoid [GPS], which is a 2-category with a fairly strict product pseudofunctor, which is strictly associative and unital. The extra data needed for a the monoidal bicategory to be symmetric is carried across the equivalence but does not necessarily get any stricter. It is unlikely that a symmetric monoidal bicategory can be strictified any further.

The symmetric monoidal structure on a bicategory translates into an H -space structure on its classifying space. In the case of symmetric monoidal categories, we know that we actually obtain an infinite loop space structure upon completion. To show that this is also the case for (strict) symmetric monoidal bicategories we will be using Segal's Γ -space machine in the context of bicategories.

Let $Bicat$ denote the category of (small) bicategories and pseudofunctors.

Definition 3.3. A pseudofunctor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ is an *equivalence of bicategories* if there exists a pseudofunctor $\mathcal{G} : \mathcal{D} \rightarrow \mathcal{C}$ and natural equivalences, i.e. weakly invertible transformations,

$$id_{\mathcal{C}} \simeq \mathcal{G} \circ \mathcal{F} \quad \text{and} \quad id_{\mathcal{D}} \simeq \mathcal{F} \circ \mathcal{G}.$$

Other authors use the term *biequivalence* to refer to the definition above.

Definition 3.4. A Γ -*bicategory* \mathcal{A} is a functor $\mathcal{A} : \mathbf{Fin}_* \rightarrow \mathbf{Bicat}$. We say \mathcal{A} is *special* if the map

$$P_n : \mathcal{A}(\underline{n}) \rightarrow \mathcal{A}(\underline{1})^{\times n}$$

is an equivalence of bicategories for all $n \geq 0$.

This definition is analogous to that of a special Γ -space, with the connection made clear by the following lemma, where $|\mathbf{N}(-)|$ denotes the classifying space functor as constructed in Appendix A.

Lemma 3.5. *Let \mathcal{A} be a special Γ -bicategory. Then $|\mathbf{N}\mathcal{A}| : \Gamma \rightarrow \mathbf{Top}$ is a special Γ -space.*

Proof. The classifying space functor $|\mathbf{N}(-)| : \mathbf{Bicat} \rightarrow \mathbf{Top}$ preserves products and sends equivalences of bicategories to homotopy equivalences of spaces (Proposition A.6). \square

Theorem 3.6. *Let \mathcal{M} be a (strict) symmetric monoidal bicategory. Then there is a special Γ -bicategory $\widehat{\mathcal{M}}$ such that*

$$\widehat{\mathcal{M}}(\underline{1}) \cong \mathcal{M}.$$

Therefore the classifying space $|\mathbf{N}\mathcal{M}|$ is an infinite loop space upon group completion.

We delay the proof of this theorem until Section 5. Given a (strict) symmetric monoidal bicategory \mathcal{M} , we denote the associated spectrum produced by the machine above by $\mathbb{A}(\mathcal{M})$.

We would like to compare the Γ -bicategory construction to the standard Γ -category construction. To that end, we provide a map of the corresponding Γ -spaces obtained from a symmetric monoidal bicategory and a symmetric monoidal category when these are closely related.

Let $(\mathcal{N}, \oplus, 0, \tau)$ be a permutative category enriched over simplicial spaces. We further require that for all objects A and B , the simplicial space $\mathcal{N}(A, B)$ is the levelwise nerve of a simplicial category (that is, a simplicial object in the category of small categories).

In Section 6, we construct a Γ -category $\tilde{\mathcal{N}}$ inspired by the construction of the Γ -bicategory and relying on the simplicial enrichment. This Γ -category is levelwise equivalent to the standard Γ -category construction [May1]. The following theorem makes the relationship between the two constructions explicit.

Theorem 3.7. *Let \mathcal{M} be a strict symmetric monoidal bicategory and \mathcal{N} a permutative category enriched over simplicial categories as above. Suppose that \mathcal{M} and \mathcal{N} have the same set of objects, and the symmetric monoidal structures coincide. Furthermore, suppose that for all pairs of objects (A, B) there is a map*

$$N\mathcal{M}(A, B) \rightarrow \mathcal{N}(A, B),$$

and that these maps are compatible with the categorical and symmetric monoidal structures on \mathcal{M} and \mathcal{N} . More precisely, this means that these maps commute with (horizontal) composition and the monoidal product, and send the identity 1-morphism to the identity morphism. Then there is a canonical map of Γ -spaces

$$|\mathbf{N}\widehat{\mathcal{M}}| \rightarrow |N\tilde{\mathcal{N}}|.$$

Therefore there is an induced map of spectra

$$\mathbb{A}(\mathcal{M}) \rightarrow \mathbb{E}(\mathcal{N}).$$

The proof of this theorem is in Section 7.

4. DELOOPING K -THEORY OF BIMONOIDAL CATEGORIES

In [BDR], N. Baas, B. Dundas and J. Rognes introduce the notions of $2K$ -theory and 2-vector bundles as a way to categorify topological K -theory and vector bundles. One of their objectives is to define a cohomology theory of a geometric nature that has chromatic level 2.

In general they define the K -theory space of a (symmetric) bimonoidal category \mathcal{R} in terms of a bar construction for monoidal categories (Subsection 4.1) and construct an equivalence of spaces

$$(1.1) \quad \mathcal{K}(\mathcal{R}) \xrightarrow{\sim} K(\mathbb{E}(\mathcal{R}))$$

between the K -theory of the bimonoidal category \mathcal{R} and the algebraic K -theory space of the ring spectrum $\mathbb{E}(\mathcal{R})$ (Subsection 4.2).

In Subsection 4.3 we prove that the K -theory space of \mathcal{R} can be constructed as the appropriate group completion of the classifying space the bicategory of modules over \mathcal{R} , $\mathcal{M}(\mathcal{R})$, and in Subsection 4.4 we show that $\mathcal{M}(\mathcal{R})$ is symmetric monoidal.

As such, we can use $\mathcal{M}(\mathcal{R})$ as input for Theorem 3.6 to obtain an infinite loop space structure on $\mathcal{K}(\mathcal{R})$. On the other hand, the algebraic K -theory of the ring spectrum $\mathbb{E}(\mathcal{R})$ is constructed as the classifying space of the symmetric monoidal category of modules $\mathcal{M}(\mathbb{E}(\mathcal{R}))$. With $\mathcal{M} = \mathcal{M}(\mathcal{R})$ and $\mathcal{N} = \mathcal{M}(\mathbb{E}(\mathcal{R}))$ as input for Theorem 3.7 we obtain a map of spectra

$$\mathbb{K}(\mathcal{R}) = \mathbb{A}(\mathcal{M}(\mathcal{R})) \longrightarrow \mathbb{E}(\mathcal{M}(\mathbb{E}(\mathcal{R}))) = \mathbb{K}(\mathbb{E}(\mathcal{R}))$$

which at the level of zeroth spaces gives the equivalence in (1.1).

4.1. Definition of K -theory of a bimonoidal category. A (symmetric) *bimonoidal category* $(\mathcal{R}, \oplus, \otimes)$ is a category \mathcal{R} endowed with a symmetric monoidal structure \oplus and a (symmetric) monoidal structure \otimes , which distributes over \oplus up to coherent isomorphisms. The precise definition can be found in [Lap] for the symmetric case and in [EM] for the non-symmetric one, although in the latter the distributivity isomorphisms are not required to be invertible. Any symmetric bimonoidal category is equivalent to a bipermutative category [May3, Prop. VI.3.5] and any bimonoidal category is equivalent to a strict bimonoidal category [Gui, Thm. 1.2].

Definition 4.1. A *strict bimonoidal category* $(\mathcal{R}, \oplus, 0, \gamma_{\oplus}, \otimes, 1, \delta)$ is a permutative category $(\mathcal{R}, \oplus, 0, \gamma_{\oplus})$, together with a strict monoidal structure $(\otimes, 1)$, such that right distributivity and nullity of zero hold strictly, and there is a left distributivity natural isomorphism

$$\delta : a \otimes (b \oplus c) \rightarrow (a \otimes b) \oplus (a \otimes c).$$

These satisfy the coherence axioms spelled out in [Gui, Definition 3.1].

If we further require the product \otimes to be permutative (strict symmetric monoidal), the category is called *bipermutative*. Strict bimonoidal and bipermutative categories are analogues of semirings and commutative semirings, respectively.

Let $(\mathcal{R}, \oplus, 0, c_\oplus, \otimes, 1, \delta)$ be a strict bimonoidal category. Then, as in [BDR], we can define $M_n(\mathcal{R})$, the category of $n \times n$ matrices over \mathcal{R} . Its objects are matrices $V = (V_{i,j})_{i,j=1}^n$ whose entries are objects of \mathcal{R} . The morphisms are matrices $\phi = (\phi_{i,j})_{i,j=1}^n$ of isomorphisms in \mathcal{R} , such that the source (resp, target) of $\phi_{i,j}$ is the (i,j) -entry of the source (target) of ϕ . As a category, $M_n(\mathcal{R})$ is isomorphic to $\mathcal{R}^{n \times n}$.

Moreover, $M_n(\mathcal{R})$ is a monoidal category, with multiplication

$$M_n(\mathcal{R}) \times M_n(\mathcal{R}) \xrightarrow{\cdot} M_n(\mathcal{R})$$

given by sending the pair (U, V) to

$$W_{ik} = \bigoplus_{j=1}^n U_{ij} \otimes V_{jk}.$$

Since \oplus is strictly associative, there is no ambiguity.

This multiplication has a unit object I_n , given by the matrix with 1 in the diagonal and 0 elsewhere. The objects 0 and 1 are strict units for \oplus and \otimes respectively, and the nullity of 0 holds strictly, so I_n is a strict unit as well.

Proposition 4.2. [BDR, 3.3] *Matrix multiplication makes $(M_n(\mathcal{R}), \cdot, I_n)$ into a monoidal category.*

The natural associativity isomorphism

$$\alpha : U \cdot (V \cdot W) \rightarrow (U \cdot V) \cdot W$$

is given by entry-wise use of c_\oplus and δ .

Recall that if R is a semi-ring, $GL_n(R)$ is the subgroup of $M_n(R)$ that contains all the matrices whose determinant is a unit in the ring completion $Gr_+(R)$. The following definition is also taken from [BDR].

Definition 4.3. Let $GL_n(\mathcal{R}) \subset M_n(\mathcal{R})$ be the full subcategory of matrices $V = (V_{i,j})_{i,j=1}^n$ whose matrix of path components lies in $GL_n(\pi_0(\mathcal{R}))$. We call $GL_n(\mathcal{R})$ the category of *weakly invertible matrices*. By convention we will let $GL_0(\mathcal{R}) = \underline{1}$ be the unit category, with one object and one morphism.

Note that $GL_n(\mathcal{R})$ inherits a monoidal structure from $M_n(\mathcal{R})$.

Given a monoidal category \mathcal{M} , the authors in [BDR] define a bar construction for monoidal categories, $B\mathcal{M}$, which is a simplicial object in Cat . As pointed out in Remark A.4, this definition coincides with the Segal nerve of the bicategory $\Sigma\mathcal{M}$, that is, the bicategory with one object whose category of morphisms is given by \mathcal{M} .

We note that block sum of matrices in \mathcal{R} makes

$$\coprod_{n \geq 0} |BGL_n(\mathcal{R})|$$

into an H -space, and hence we define the K -theory of \mathcal{R} , as the group completion

$$\mathcal{K}(\mathcal{R}) := \Omega B \left(\coprod_{n \geq 0} |BGL_n(\mathcal{R})| \right).$$

The motivation behind the definition of K -theory for bimonoidal categories comes from the categorification of complex K -theory. As we know well, the complex K -theory space classifies virtual vector bundles.

A 2-vector space, as defined in [KV], is a category equivalent to $(Vect_{\mathbb{C}})^n$ for some n . Heuristically, this should be thought of as a module category over $Vect_{\mathbb{C}}$. In [BDR], the authors introduce the notion of a complex 2-vector bundle over a topological space and construct a classifying space for these bundles. A 2-vector bundle is roughly a bundle of 2-vector spaces over X , defined in terms of transition functions, which are given by matrices of vector spaces. For the precise definition we refer the reader to [BDR, Section 2].

One of the main results in [BDR] is that the stable equivalence classes of virtual 2-vector bundles over a space X are in one-to-one correspondence with homotopy classes of maps from X to $\mathcal{K}(Vect_{\mathbb{C}})$, where $Vect_{\mathbb{C}}$ is considered as a bipermutative category using direct sum and tensor product.

4.2. Relationship with algebraic K -theory. Given a strict bimonoidal category \mathcal{R} , by forgetting the multiplicative structure, we can construct the spectrum $\mathbb{E}(\mathcal{R})$ associated to the permutative category (\mathcal{R}, \oplus) . The results of [EM] and [May2] show that the multiplicative structure of \mathcal{R} makes $\mathbb{E}(\mathcal{R})$ into a ring spectrum, and furthermore, if \mathcal{R} is bipermutative, $\mathbb{E}(\mathcal{R})$ is an E_{∞} ring spectrum. Note that in [BDR, BDRR] the authors denote $\mathbb{E}(\mathcal{R})$ by $H\mathcal{R}$, pointing out the analogy to the Eilenberg-MacLane spectrum of a ring.

The natural inclusion $B\mathcal{R} \rightarrow \mathbb{E}_0(\mathcal{R})$ extends to a map $|BGL_n(\mathcal{R})| \rightarrow BGL_n(\mathbb{E}(\mathcal{R}))$. Taking the group completion of the disjoint union over n , we get the map

$$(4.4) \quad \mathcal{K}(\mathcal{R}) \xrightarrow{\sim} K(\mathbb{E}(\mathcal{R})),$$

of (1.1), which [BDRR] proves to be an equivalence of spaces. Here the right-hand side is the zeroth space of $\mathbb{K}(\mathbb{E}(\mathcal{R}))$, which we view as the algebraic K -theory ring spectrum associated to the ring spectrum $\mathbb{E}(\mathcal{R})$.

We are headed towards an intrinsic description of the left-hand side as an infinite loop space in Theorem 4.8 and a proof that the equivalence is compatible with these infinite loop space structures in Corollary 4.10.

4.3. K -theory as a classifying space of a bicategory.

Definition 4.5. Let $\mathcal{M}(\mathcal{R})$ be the bicategory of finite dimensional free modules over \mathcal{R} , defined as follows. The objects are labeled by the natural numbers $\mathbf{n} \geq 0$. Given objects \mathbf{n}, \mathbf{m} , the category of morphisms is

$$\mathcal{M}(\mathcal{R})(\mathbf{n}, \mathbf{m}) = \begin{cases} GL_n(\mathcal{R}) & \text{if } n = m \\ \emptyset & \text{if } n \neq m. \end{cases}$$

and the composition is given by matrix multiplication. In other words,

$$\mathcal{M}(\mathcal{R}) = \coprod_{n \geq 0} \Sigma GL_n(\mathcal{R}).$$

Example 4.6. Let $Vect_k$ be the bipermutative category of vector spaces over the field k . Then Mod_{Vect_k} is a sub-bicategory of the bicategory of 2-vector spaces defined by Kapranov and Voevodsky [KV]. The 1-morphisms are matrices of vector spaces such that their matrices of dimensions have determinant ± 1 .

We can use the bicategory $\mathcal{M}(\mathcal{R})$ to give an alternative definition of the K -theory of \mathcal{R} . We have that

$$\coprod_{n \geq 0} |BGL_n(\mathcal{R})| = \coprod_{n \geq 0} \mathbf{N} \Sigma GL_n(\mathcal{R}) = \mathbf{N} \left(\coprod_{n \geq 0} \Sigma GL_n(\mathcal{R}) \right) = \mathbf{N} \mathcal{M}(\mathcal{R}),$$

where $\mathbf{N}(-)$ denotes the Segal nerve (see Appendix A). Hence, we can describe the K -theory space as $\Omega B|\mathbf{N}\mathcal{M}(\mathcal{R})|$.

This is the definition we will use in the following sections. Furthermore, we will show that the H -space structure comes from a pseudofunctor

$$\mathcal{M}(\mathcal{R}) \times \mathcal{M}(\mathcal{R}) \rightarrow \mathcal{M}(\mathcal{R}),$$

which will give $\mathcal{M}(\mathcal{R})$ the structure of a symmetric monoidal bicategory.

4.4. Symmetric monoidal structure on $\mathcal{M}(\mathcal{R})$. Just as we can take direct sum of modules over a ring, we can take direct sum of modules over a bimonoidal category. This will provide $\mathcal{M}(\mathcal{R})$ with a symmetric monoidal structure, which in turn will give rise to an infinite delooping of $\mathcal{K}(\mathcal{R})$.

Theorem 4.7. *The bicategory $\mathcal{M}(\mathcal{R})$ is strict symmetric monoidal with the monoidal operation given by block sum of matrices:*

$$\begin{aligned} \boxplus : \mathcal{M}(\mathcal{R}) \times \mathcal{M}(\mathcal{R}) &\rightarrow \mathcal{M}(\mathcal{R}) \\ (\mathbf{n}, \mathbf{m}) &\mapsto \mathbf{n} + \mathbf{m} \\ (U, V) &\mapsto \left[\begin{array}{c|c} U & 0 \\ \hline 0 & V \end{array} \right] \\ (\varphi, \psi) &\mapsto \left[\begin{array}{c|c} \varphi & 0 \\ \hline 0 & \psi \end{array} \right]. \end{aligned}$$

The matrix $[0]$ is the matrix with all entries equal to 0, the unit of \oplus in \mathcal{R} .

Proof. We first note that the operation described above gives a strict pseudofunctor of bicategories, since it preserves the identity and the composition:

$$\begin{aligned} I_n \boxplus I_m &= I_{n+m}, \\ \left[\begin{array}{c|c} U' & 0 \\ \hline 0 & V' \end{array} \right] * \left[\begin{array}{c|c} U & 0 \\ \hline 0 & V \end{array} \right] &= \left[\begin{array}{c|c} U' * U & 0 \\ \hline 0 & V' * V \end{array} \right]. \end{aligned}$$

The second equation holds because of the strict nullity and unity of 0 in \mathcal{R} .

The unit of \boxplus is $\mathbf{0}$. We note that for $U \in GL_n(\mathcal{R})$, $V \in GL_m(\mathcal{R})$, and $W \in GL_p(\mathcal{R})$:

$$\begin{aligned} (U \boxplus V) \boxplus W &= U \boxplus (V \boxplus W), \\ I_0 \boxplus U &= U = U \boxplus I_0. \end{aligned}$$

The natural equivalence $\beta_{n,m} : \mathbf{n} \boxplus \mathbf{m} \rightarrow \mathbf{m} \boxplus \mathbf{n}$ is given by the block matrix

$$\left[\begin{array}{c|c} 0 & I_m \\ \hline I_n & 0 \end{array} \right].$$

Since 0 and 1 are strict units in \mathcal{R} , for $U \in GL_n(\mathcal{R})$ and $V \in GL_m(\mathcal{R})$,

$$\beta_{n,m} * (U \boxplus V) = \left[\begin{array}{c|c} 0 & V \\ \hline U & 0 \end{array} \right] = (V \boxplus U) * \beta_{n,m},$$

so β is a strict transformation.

We note that $\beta_{m,n} * \beta_{n,m} = I_{n+m}$ which both implies that β is a natural isomorphism and that it is its self-inverse. We conclude that $\mathcal{M}(\mathcal{R})$ is a strict symmetric monoidal bicategory. \square

This, together with Theorem 3.6 proves the following theorem.

Theorem 4.8. *The K -theory space of the bimonoidal category \mathcal{R} is an infinite loop space, with the additive structure provided by the block sum of matrices. More precisely, we can identify $\mathcal{K}(\mathcal{R})$ with the zeroth space of the spectrum $\mathbb{K}(\mathcal{R}) := \mathbb{A}(\mathcal{M}(\mathcal{R}))$.*

Now we can look back at equation (4.4). The right hand side is an infinite loop space, with the structure given by the Γ -space construction on the classifying space of the symmetric monoidal category $\mathcal{M}(\mathbb{E}(\mathcal{R}))$ of finitely generated free modules over $\mathbb{E}(\mathcal{R})$. In order to prove that the map in equation (4.4) is a map of infinite loop spaces we will prove that the map extends to a map of Γ -spaces. In particular, we prove the following theorem.

Theorem 4.9. *There is a zigzag of maps of Γ -spaces*

$$|\widehat{\mathbf{N}\mathcal{M}(\mathcal{R})}| \longrightarrow |\widehat{N\mathcal{M}(\mathbb{E}(\mathcal{R}))}| \xleftarrow{\sim} |\widehat{N\mathcal{M}(\mathbb{E}(\mathcal{R}))}|.$$

At level 1, the right-hand map is an equality, and the left-hand map corresponds to the map in equation (4.4).

Corollary 4.10. *There is a zigzag of equivalences of spectra*

$$\mathbb{K}(\mathcal{R}) \xrightarrow{\sim} \tilde{\mathbb{K}}(\mathbb{E}(\mathcal{R})) \xleftarrow{\sim} \mathbb{K}(\mathbb{E}(\mathcal{R})),$$

which at the level of zeroth spaces gives the maps

$$\mathcal{K}(\mathcal{R}) \xrightarrow[(4.4)]{} \tilde{K}(\mathbb{E}(\mathcal{R})) \xleftarrow{=} K(\mathbb{E}(\mathcal{R})).$$

The model we are taking for $\mathbb{E}(\mathcal{R})$ is that of [EM]. We construct $\mathcal{M}(\mathbb{E}(\mathcal{R}))$ as follows.

Let $GL_n(\mathbb{E}(\mathcal{R}))$ be the group-like monoid of weakly invertible matrices over $\mathbb{E}(\mathcal{R})$. It is defined by the pullback

$$\begin{array}{ccc} GL_n(\mathbb{E}(\mathcal{R})) & \longrightarrow & \operatorname{hocolim}_{\mathbf{m} \in I} \Omega^m M_n(\mathbb{E}(\mathcal{R}))(m) \\ \downarrow & & \downarrow \\ GL_n(\pi_0 \mathbb{E}(\mathcal{R})) & \longrightarrow & M_n(\pi_0 \mathbb{E}(\mathcal{R})), \end{array}$$

The category $\mathcal{M}(\mathbb{E}(\mathcal{R}))$ has as objects the natural numbers \mathbf{n} . The space of morphisms is given by

$$\mathcal{M}(\mathbb{E}(\mathcal{R}))(\mathbf{n}, \mathbf{m}) = \begin{cases} GL_n(\mathbb{E}(\mathcal{R})) & \text{if } n = m \\ \emptyset & \text{if } n \neq m. \end{cases}$$

Note that since $\mathbb{E}(\mathcal{R})(0) = N\mathcal{R}$, there is a map of spaces

$$NGL_n(\mathcal{R}) \longrightarrow GL_n(\mathbb{E}(\mathcal{R})).$$

Proof of Theorem 4.9. In the construction of the ring spectrum $\mathbb{E}(\mathcal{R})$ in [EM], the spaces of the spectrum are nerves of simplicially enriched categories. This implies that $GL_n(\mathbb{E}(\mathcal{R}))$ is the nerve of a simplicially enriched category, and thus also of a simplicial category. Thus $\mathcal{M}(\mathbb{E}(\mathcal{R}))$ satisfies the conditions for the construction and theorems of Sections 6 and 3.7.

Furthermore, since the map $NGL_n(\mathcal{R}) \rightarrow GL_n(\mathbb{E}(\mathcal{R}))$ is compatible with the block sum of matrices, we can let \mathcal{M} and \mathcal{N} be $\mathcal{M}(\mathcal{R})$ and $\mathcal{M}(\mathbb{E}(\mathcal{R}))$ respectively in the statement of Theorem 3.7. This proves the theorem. \square

5. PROOF OF THEOREM 3.6

We will first construct the bicategory $\widehat{\mathcal{M}}(\underline{n})$ for $n \geq 0$ as follows:

- (1) Objects are of the form $\{A_S, a_{S,T}\}_{S,T}$, where S runs over all the subsets of \underline{n} that do not contain the basepoint 0; (S, T) runs over all pairs of such subsets such that $S \cap T = \emptyset$; $A_S \in \text{Ob}\mathcal{M}$ and $a_{S,T} : A_{S \cup T} \rightarrow A_S \boxplus A_T$ is a 1-equivalence, that is a 1-morphism that is invertible up to isomorphism. We require further

- (a) $A_\emptyset = 0$;
- (b) $a_{\emptyset, S} = I_{A_S} = a_{S, \emptyset}$;
- (c) for every triple (S, T, U) of subsets such that $S \cap T = S \cap U = T \cap U = \emptyset$, the diagram

$$(5.1) \quad \begin{array}{ccc} A_{S \cup T \cup U} & \xrightarrow{a_{S, T \cup U}} & A_S \boxplus A_{T \cup U} \\ \downarrow a_{S \cup T, U} & & \downarrow I_{A_S} \boxplus a_{T, U} \\ A_{S \cup T} \boxplus A_U & \xrightarrow{a_{S, T} \boxplus I_{A_U}} & A_S \boxplus A_T \boxplus A_U \end{array}$$

strictly commutes;

- (d) for every pair of subsets (S, T) , the diagram

$$(5.2) \quad \begin{array}{ccc} A_{S \cup T} & \xrightarrow{a_{S, T}} & A_S \boxplus A_T \\ \parallel & & \downarrow \beta_{A_S, A_T} \\ A_{T \cup S} & \xrightarrow{a_{T, S}} & A_T \boxplus A_S \end{array}$$

strictly commutes.

- (2) A 1-morphism between $\{A_S, a_{S,T}\}$ and $\{A'_S, a'_{S,T}\}$ is given by a system $\{f_S, \phi_{S,T}\}_{S,T}$, where S, T are as above; $f_S : A_S \rightarrow A'_S$ is a 1-morphism in \mathcal{M} and $\phi_{S,T}$ is a 2-isomorphism:

$$\begin{array}{ccc} A_{S \cup T} & \xrightarrow{a_{S, T}} & A_S \boxplus A_T \\ f_{S \cup T} \downarrow & \not\Downarrow \phi_{S, T} & \downarrow f_S \boxplus f_T \\ A'_{S \cup T} & \xrightarrow{a'_{S, T}} & A'_S \boxplus A'_T. \end{array}$$

We require:

- (a) $\phi_{\emptyset, S} : f_S * I_{A_S} = f_S \Rightarrow f_S = I_{A_S} * f_S$ is the identity 2-morphism and similarly for $\phi_{S, \emptyset}$;

(b) for every pairwise disjoint S, T, U the following equation holds

$$(5.3) \quad \begin{array}{ccccc} A_{S \cup T \cup U} & \xrightarrow{a_{S,T \cup U}} & A_S \boxplus A_{T \cup U} & \xrightarrow{I_{A_S} \boxplus a_{T,U}} & A_S \boxplus A_T \boxplus A_U \\ \downarrow f_{S \cup T \cup U} & \searrow \not\phi_{S,T \cup U} & \downarrow f_S \boxplus f_{T \cup U} & \searrow \not l^{-1} * r \boxplus \phi & \downarrow f_S \boxplus f_T \boxplus f_U \\ A'_{S \cup T \cup U} & \xrightarrow{a'_{S,T \cup U}} & A'_S \boxplus A'_{T \cup U} & \xrightarrow{I_{A'_S} \boxplus a'_{T,U}} & A'_S \boxplus A'_T \boxplus A'_U \\ & & \parallel & & \\ A_{S \cup T \cup U} & \xrightarrow{a_{S,T \cup U}} & A_{S \cup T} \boxplus A_U & \xrightarrow{a_{S,T} \boxplus I_{A_U}} & A_S \boxplus A_T \boxplus A_U \\ \downarrow f_{S \cup T \cup U} & \searrow \not\phi_{S \cup T,U} & \downarrow f_{S \cup T} \boxplus f_U & \searrow \not\phi \boxplus l^{-1} * r & \downarrow f_S \boxplus f_T \boxplus f_U \\ A'_{S \cup T \cup U} & \xrightarrow{a'_{S,T \cup U}} & A'_{S \cup T} \boxplus A'_U & \xrightarrow{a'_{S,T} \boxplus I_{A'_U}} & A'_S \boxplus A'_T \boxplus A'_U, \end{array}$$

where r, l denote the coherent identity isomorphisms in \mathcal{M} , that is $l_f : I_B * f \rightarrow f$ and $r_f : f * I_A \rightarrow f$;

(c) for every S, T the following equation holds:

$$(5.4) \quad \begin{array}{ccccc} A_{S \cup T} & \xrightarrow{a_{S,T}} & A_S \boxplus A_T & \xrightarrow{\beta} & A_T \boxplus A_S \\ \downarrow f_{S \cup T} & \searrow \not\phi_{S,T} & \downarrow f_S \boxplus f_T & \searrow \not\beta_{f_S, f_T} & \downarrow f_T \boxplus f_S \\ A'_{S \cup T} & \xrightarrow{a'_{S,T}} & A'_S \boxplus A'_T & \xrightarrow{\beta} & A'_T \boxplus A'_S \end{array} = \begin{array}{ccccc} A_{T \cup S} & \xrightarrow{a_{T,S}} & A_T \boxplus A_S \\ \downarrow f_{T \cup S} & \searrow \not\phi_{T,S} & \downarrow f_T \boxplus f_S \\ A'_{T \cup S} & \xrightarrow{a'_{T,S}} & A'_T \boxplus A'_S. \end{array}$$

(3) Given 1-morphisms $\{f_S, \phi_{S,T}\}, \{g_S, \gamma_{S,T}\} : \{A_S, a_{S,T}\} \rightarrow \{A'_S, a'_{S,T}\}$, a 2-morphism between them is given by a system $\{\psi_S\}$ of 2-morphisms in \mathcal{M} , $\psi_S : f_S \Rightarrow g_S$, such that for all S, T as above the following equation holds:

$$(5.5) \quad \begin{array}{ccc} \begin{array}{ccc} A_{S \cup T} & \xrightarrow{a_{S,T}} & A_S \boxplus A_T \\ \downarrow g_{S \cup T} & \searrow \not\psi_{S \cup T} & \downarrow f_S \boxplus f_T \\ A'_{S \cup T} & \xrightarrow{a'_{S,T}} & A'_S \boxplus A'_T \end{array} & = & \begin{array}{ccc} A_{S \cup T} & \xrightarrow{a_{S,T}} & A_S \boxplus A_T \\ \downarrow g_{S \cup T} & \searrow \not\gamma_{S,T} & \downarrow g_S \boxplus g_T \\ A'_{S \cup T} & \xrightarrow{a'_{S,T}} & A'_S \boxplus A'_T \end{array} \end{array}$$

We now need to show that these data indeed define a bicategory. We will first show that given objects $\{A_S, a_{S,T}\}, \{A'_S, a'_{S,T}\}$, the 1-morphisms and 2-morphisms form a category $\widehat{\mathcal{M}}(\underline{n})(\{A_S, a_{S,T}\}, \{A'_S, a'_{S,T}\})$.

Vertical composition of 2-morphisms $\{\psi_S\}, \{\psi'_S\}$ is defined componentwise. We show that this composition satisfies equation (5.5). Given

$$\begin{array}{ccc} & \xrightarrow{\{f, \phi\}} & \\ & \Downarrow \{\psi\} & \\ \{A_S, a_{S,T}\} & \xrightarrow{\{g, \gamma\}} & \{A'_S, a'_{S,T}\} \\ & \Downarrow \{\psi'\} & \\ & \xrightarrow{\{h, \eta\}} & \end{array}$$

we see that for all S, T :

$$\begin{array}{ccc} \begin{array}{ccc} A_{S \cup T} & \xrightarrow{a_{S,T}} & A_S \boxplus A_T \\ \downarrow h & \searrow f \boxplus f & \downarrow f \boxplus f \\ \begin{array}{c} \xleftarrow{\psi'} \\ \downarrow g \\ \xleftarrow{\psi} \end{array} & \xrightarrow{f \boxplus f} & A'_S \boxplus A'_T \\ A'_{S \cup T} & \xrightarrow{a'_{S,T}} & \end{array} & = & \begin{array}{ccc} A_{S \cup T} & \xrightarrow{a_{S,T}} & A_S \boxplus A_T \\ \downarrow h & \searrow g \boxplus g & \downarrow g \boxplus g \\ \begin{array}{c} \xleftarrow{\psi'} \\ \downarrow g \\ \xleftarrow{\psi} \end{array} & \xrightarrow{g \boxplus g} & A'_S \boxplus A'_T \\ A'_{S \cup T} & \xrightarrow{a'_{S,T}} & \end{array} \\ \\ \begin{array}{ccc} A_{S \cup T} & \xrightarrow{a_{S,T}} & A_S \boxplus A_T \\ \downarrow h & \searrow h \boxplus h & \downarrow h \boxplus h \\ \begin{array}{c} \xleftarrow{\psi'} \\ \downarrow g \boxplus g \\ \xleftarrow{\psi} \end{array} & \xrightarrow{h \boxplus h} & A'_S \boxplus A'_T \\ A'_{S \cup T} & \xrightarrow{a'_{S,T}} & \end{array} & = & \begin{array}{ccc} A_{S \cup T} & \xrightarrow{a_{S,T}} & A_S \boxplus A_T \\ \downarrow h & \searrow h \boxplus h & \downarrow h \boxplus h \\ \begin{array}{c} \xleftarrow{\psi'} \\ \downarrow g \boxplus g \\ \xleftarrow{\psi} \end{array} & \xrightarrow{h \boxplus h} & A'_S \boxplus A'_T \\ A'_{S \cup T} & \xrightarrow{a'_{S,T}} & \end{array} \end{array}$$

as wanted.

We also note that $\{\text{id}_{f_S}\}$ is a well-defined automorphism for $\{f_S, \phi_{S,T}\}$ and it is the identity of the componentwise composition.

The composition functor $*$ is given by:

$$\begin{aligned} (\{g_S, \gamma_{S,T}\}, \{f_S, \phi_{S,T}\}) &\mapsto \{g_S * f_S, (\gamma \diamond \phi)_{S,T}\} \\ (\{\psi'_S\}, \{\psi_S\}) &\mapsto \{\psi'_S * \psi_S\}, \end{aligned}$$

where the 2-morphism $(\gamma \diamond \phi)_{S,T}$ is defined by the pasting diagram:

$$\begin{array}{ccc} A_{S \cup T} & \xrightarrow{a_{S,T}} & A_S \boxplus A_T \\ \downarrow f_{S \cup T} & \searrow \not\llcorner \phi_{S,T} & \downarrow f_S \boxplus f_T \\ A'_{S \cup T} & \xrightarrow{a'_{S,T}} & A'_S \boxplus A'_T \\ \downarrow g_{S \cup T} & \searrow \not\llcorner \gamma_{S,T} & \downarrow g_S \boxplus g_T \\ A''_{S \cup T} & \xrightarrow{a''_{S,T}} & A''_S \boxplus A''_T \end{array}$$

$g_S * f_S \boxplus g_T * f_T$

The unmarked 2-isomorphism is the coherent 2-morphism corresponding to the weak functoriality of \boxplus .

Showing that $\{g_S * f_S, (\gamma \diamond \phi)_{S,T}\}$ is a well-defined 1-morphism (that is, it satisfies equations (5.3) and (5.4)) can be done again using pasting diagrams, the coherence of the functoriality isomorphism and the fact that both $\{f_S, \phi_{S,T}\}$ and $\{g_S, \gamma_{S,T}\}$ satisfy those same equations. Analogously we can show that $\{\psi'_S * \psi_S\}$ is a well-defined 2-morphism.

The natural associativity isomorphism in this bicategory is given by the componentwise associativity isomorphisms in \mathcal{M} . More precisely, given composable 1-morphisms $\{f_S, \phi_{S,T}\}$, $\{g_S, \gamma_{S,T}\}$, and $\{h_S, \eta_{S,T}\}$, we define the 2-morphism $\{\alpha_S\}$, where

$$\alpha_S : h_S * (g_S * f_S) \Rightarrow (h_S * g_S) * f_S$$

is the associativity isomorphism in \mathcal{M} .

The fact that

$$\{(h_S * g_S) * f_S, ((\eta \diamond \gamma) \diamond \phi)_{S,T}\} \Rightarrow \{h_S * (g_S * f_S), (\eta \diamond (\gamma \diamond \phi))_{S,T}\}$$

is an allowed 2-morphism in $\widehat{\mathcal{M}}(\underline{n})$ will follow from the uniqueness of pasting diagrams and the functoriality of \boxplus . Naturality and the pentagonal axiom follow from those in \mathcal{M} .

Given an object $\{A_S, a_{S,T}\}$, the identity 1-morphism is given by $\{I_{A_S}, \iota_{S,T}\}$, where ι is the appropriate coherent 2-isomorphism obtained by the composition of instances of the 2-morphisms \boxplus^0 and $r^{-1} * l$. It is clear that this is an allowed 1-morphism in $\widehat{\mathcal{M}}(\underline{n})$ and that it is a weak identity, with right and left identity 2-isomorphisms given by $\{r_{f_S}\}$, $\{l_{f_S}\}$. We conclude thus that $\widehat{\mathcal{M}}(\underline{n})$ is indeed a bicategory.

We now need to prove that this construction extends to a functor $\widehat{\mathcal{M}} : \text{Fin}_* \rightarrow \text{Bicat}$. Given a morphism $\theta : \underline{n} \rightarrow \underline{m}$ in Fin_* we define a pseudofunctor

$$\theta_* : \widehat{\mathcal{M}}(\underline{n}) \rightarrow \widehat{\mathcal{M}}(\underline{m})$$

as follows:

$$\begin{aligned} \{A_S, a_{S,T}\} &\longmapsto \{A_U^\theta, a_{U,V}^\theta\} = \{A_{\theta^{-1}(U)}, a_{\theta^{-1}(U), \theta^{-1}(V)}\} \\ \{f_S, \phi_{S,T}\} &\longmapsto \{f_U^\theta, \phi_{U,V}^\theta\} = \{f_{\theta^{-1}(U)}, \phi_{\theta^{-1}(U), \theta^{-1}(V)}\} \\ \{\psi_S\} &\longmapsto \{\psi_U^\theta\} = \{\psi_{\theta^{-1}(U)}\}, \end{aligned}$$

where U, V range over disjoint subsets of \underline{m} that do not contain the basepoint. Since θ is basepoint preserving, $\theta^{-1}(U)$ does not contain the basepoint and it is an allowed indexing subset of \underline{n} . Also, since U and V are disjoint, their pre-images under θ are also disjoint.

This assignment commutes strictly with all the compositions and identities in $\widehat{\mathcal{M}}(\underline{n})$ and $\widehat{\mathcal{M}}(\underline{m})$, giving a pseudofunctor between these bicategories.

It is clear from the construction that $\widehat{\mathcal{M}}(\underline{1})$ is isomorphic to \mathcal{M} .

We will end the proof by showing that for every $n \geq 0$, the pseudofunctor

$$p_n : \widehat{\mathcal{M}}(\underline{n}) \rightarrow \mathcal{M}^{\times n}$$

is an equivalence of bicategories (Definition 3.3). This will show that the Γ -bicategory is special. For ease of notation we will denote the subset $\{i\} \in \underline{n}$ as i . The pseudofunctor p_n

takes

$$\begin{aligned}\{A_S, a_{S,T}\} &\longmapsto \{A_i\}_{i=1}^n \\ \{f_S, \phi_{S,T}\} &\longmapsto \{f_i\}_{i=1}^n \\ \{\psi_S\} &\longmapsto \{\psi_i\}_{i=1}^n.\end{aligned}$$

We will define an inverse pseudofunctor $i_n : \mathcal{M}^{\times n} \rightarrow \widehat{\mathcal{M}}(\underline{n})$:

$$\begin{aligned}\{A_i\}_{i=1}^n &\longmapsto \{\bigoplus_{i \in S} A_i, e_{S,T}\} \\ \{f_i\}_{i=1}^n &\longmapsto \{\bigoplus_{i \in S} f_i, \epsilon_{S,T}\} \\ \{\psi_i\}_{i=1}^n &\longmapsto \{\bigoplus_{i \in S} \psi_i\}.\end{aligned}$$

Here, $\bigoplus_{i \in S}$ denotes the iterated monoidal operation \boxplus with the usual order of the indices in $S \subset \underline{n}$. Recall that \boxplus is strictly associative,

The 1-morphism

$$e_{S,T} : \bigoplus_{i \in S \cup T} A_i \longrightarrow \bigoplus_{i \in S} A_i \boxplus \bigoplus_{i \in T} A_i$$

is the unique composition of instances of the braiding β that reorders the summands. It is clear that $\{\bigoplus_{i \in S} A_i, e_{S,T}\}$ satisfies equations (5.1) and (5.2).

The 2-isomorphism

$$\epsilon_{S,T} : (\bigoplus_{i \in S} f_i \boxplus \bigoplus_{i \in T} f_i) * e_{S,T} \rightarrow e_{S,T} * (\bigoplus_{i \in S \cup T} f_i)$$

is given by pasting the appropriate instances of β_{f_i, f_j}^2 , that is, the coherent 2-isomorphism of the braiding transformation (see [Lei, Section 1.2]). The coherence condition for these morphisms implies that the collection $\{\bigoplus_{i \in S} f_i, \epsilon_{S,T}\}$ satisfies equations (5.3) and (5.4). It is also automatic that for any $\{\psi_i\} : \{f_i\} \Rightarrow \{g_i\}$, we get that $\{\bigoplus_{i \in S} \psi_i\}$ is an allowed 2-morphism between $\{\bigoplus_{i \in S} f_i, \text{id}\}$ and $\{\bigoplus_{i \in S} g_i, \text{id}\}$.

This assignment gives a pseudofunctor with

$$i_n^2 : i_n(\{g_i\}) * i_n(\{f_i\}) \rightarrow i_n(\{g_i * f_i\})$$

and

$$i_n^0 : I_{i_n(\{A_i\})} \rightarrow i_n(\{I_{A_i}\})$$

given by the appropriate composition of instances of \boxplus^2 and \boxplus^0 respectively.

Clearly $p_n \circ i_n = \text{Id}_{\mathcal{M}^{\times n}}$. We now construct a natural equivalence

$$\xi : \text{Id}_{\widehat{\mathcal{M}}(\underline{n})} \rightarrow i_n \circ p_n.$$

Recall that a transformation η between pseudofunctors $F, G : \mathcal{C} \rightarrow \mathcal{D}$ consists of a 1-morphism $\eta_A : \mathcal{F}A \rightarrow \mathcal{G}A$ for each $A \in \text{Ob}(\mathcal{C})$, and for every pair A, B , natural 2-isomorphisms

$$\eta_f^2 : \mathcal{G}f * \eta_A \Rightarrow \eta_B * \mathcal{F}f.$$

The latter must satisfy some coherence conditions (see [Lei, Section 1.2]). A transformation is a natural equivalence if and only if the 1-morphism corresponding to each object is a 1-equivalence.

Hence, to construct the natural equivalence ξ , we need a 1-equivalence

$$\xi_{\{A_S, a_{S,T}\}} : \{A_S, a_{S,T}\} \rightarrow \left\{ \bigoplus_{i \in S} A_i, e_{S,T} \right\}$$

for every object $\{A_S, a_{S,T}\}$ in $\widehat{\mathcal{M}}(\underline{n})$.

Given the subset S , we define a^S inductively as the composition:

$$A_S \xrightarrow{a_{j,S-j}} A_j \boxplus A_{S-j} \xrightarrow{\text{id}_{A_j} \boxplus a^{S-j}} A_j \boxplus \bigoplus_{i \in S-j} A_i = \bigoplus_{i \in S} A_i,$$

where j is the smallest index in S .

Note that by conditions (5.1) and (5.2) on the $a_{S,T}$, the two compositions in the diagram below differ by a specified associativity 2-isomorphism:

$$\begin{array}{ccc} A_{S \cup T} & \xrightarrow{a_{S,T}} & A_S \boxplus A_T \\ \downarrow a^{S \cup T} & \not\Downarrow \eta_{S,T} & \downarrow a^S \boxplus a^T \\ \bigoplus_{i \in S \cup T} A_i & \xrightarrow{e_{S,T}} & \bigoplus_{i \in S} A_i \boxplus \bigoplus_{i \in T} A_i. \end{array}$$

Since associativity isomorphisms are unique, $\{a^S, \eta_{S,T}\}$ is a well-defined 1-morphism in $\widehat{\mathcal{M}}(\underline{n})$. This will be the corresponding 1-morphism of the transformation ξ .

To complete the data of the transformation, for every pair of objects $\{A_S, a_{S,T}\}, \{A'_S, a'_{S,T}\}$ in $\widehat{\mathcal{M}}(\underline{n})$ we need to provide a natural isomorphism ξ^2 , which on the component $\{f_S, \phi_{S,T}\}$ is given by a 2-morphism

$$\xi^2(\{f_S, \phi_{S,T}\}) : \left\{ \bigoplus_{i \in S} f_i, \epsilon_{S,T} \right\} * \{a^S, \eta_{S,T}\} \Rightarrow \{a'^S, \eta'_{S,T}\} * \{f_S, \phi_{S,T}\}.$$

Given S , we define a 2-isomorphism in \mathcal{M} , $\phi^S : (\bigoplus_{i \in S} f_i) * a^S \Rightarrow a'^S * f_S$, inductively as the pasting diagram:

$$\begin{array}{ccccc} A_S & \xrightarrow{a_{j,S-j}} & A_j \boxplus A_{S-j} & \xrightarrow{I_{A_j} \boxplus a'^{S-j}} & \bigoplus_{i \in S} A_i \\ \downarrow f_S & \not\Downarrow \phi_{j,S-j} & \downarrow f_j \boxplus f_{S-j} & \not\Downarrow 1 * r \boxplus \phi^{S-j} & \downarrow \bigoplus_{i \in S} f_i \\ A'_S & \xrightarrow{a'_{j,S-j}} & A'_j \boxplus A'_{S-j} & \xrightarrow{I_{A'_j} \boxplus a'^{S-j}} & \bigoplus_{i \in S} A'_i \end{array}$$

where j is the smallest index in S . We need to show that $\{\phi^S\}_S$ gives a 2-morphism in $\widehat{\mathcal{M}}(\underline{n})$, that is, it satisfies equation (5.5). This is done by induction on $|S \cup T|$ using pasting diagrams. We let $\xi^2(\{f_S, \phi_{S,T}\}) = \{\phi^S\}$.

To show the naturality of ξ^2 , we need to show that

$$\begin{array}{ccc}
 A_S & \xrightarrow{a^S} & \bigoplus_{i \in S} A_i \\
 \downarrow g_S & \swarrow \psi_S & \downarrow \bigoplus_{i \in S} f_i \\
 A'_S & \xrightarrow{a'^S} & \bigoplus_{i \in S} A'_i
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 A_S & \xrightarrow{a^S} & \bigoplus_{i \in S} A_i \\
 \downarrow g_S & \swarrow \gamma_S & \downarrow \bigoplus_{i \in S} g_i \\
 A'_S & \xrightarrow{a'^S} & \bigoplus_{i \in S} A'_i
 \end{array}$$

$\Downarrow \phi^S$ $\Downarrow \psi^S$

This follows by induction on $|S|$, using the inductive definition of ϕ^S and equation (5.5). Since ϕ^S is invertible, we get a natural isomorphism as wanted.

For the first and second axioms of a transformation [Lei, Section 1.2] we need to show

$$\begin{array}{ccc}
 A_S & \xrightarrow{a^S} & \bigoplus_{i \in S} A_i \\
 \downarrow f_S & \swarrow \phi^S & \downarrow \bigoplus_{i \in S} f_i \\
 A'_S & \xrightarrow{a'^S} & \bigoplus_{i \in S} A'_i \\
 \downarrow g_S & \swarrow \gamma^S & \downarrow \bigoplus_{i \in S} g_i \\
 A''_S & \xrightarrow{a''^S} & \bigoplus_{i \in S} A''_i
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 A_S & \xrightarrow{a^S} & \bigoplus_{i \in S} A_i \\
 \downarrow g_S * f_S & \swarrow (\gamma \circ \phi)^S & \downarrow \bigoplus_{i \in S} g_i * f_i \\
 A''_S & \xrightarrow{a''^S} & \bigoplus_{i \in S} A''_i
 \end{array}$$

and

$$\begin{array}{ccc}
 A_S & \xrightarrow{a^S} & \bigoplus_{i \in S} A_i \\
 \downarrow I_{A_S} & \swarrow \iota^S & \downarrow \bigoplus_{i \in S} I_{A_i} \\
 A_S & \xrightarrow{a^S} & \bigoplus_{i \in S} A_i
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 A_S & \xrightarrow{a^S} & \bigoplus_{i \in S} A_i \\
 \downarrow I_{A_S} & \swarrow r^{-1} & \downarrow \bigoplus_{i \in S} I_{A_i} \\
 A_S & \xrightarrow{a^S} & \bigoplus_{i \in S} A_i
 \end{array}$$

The first one is straightforward using induction on $|S|$ and the definition of $\gamma \circ \phi$. The second one holds again by induction on $|S|$, the fact that ι is a composition of instances of \boxplus^0 , r and l , and the coherence conditions of \boxplus^0 with respect to r and l .

Hence we have a natural equivalence between $Id_{\widehat{\mathcal{M}}(\underline{n})}$ and $i_n \circ p_n$. We conclude that the bicategories $\widehat{\mathcal{M}}(\underline{n})$ and $\mathcal{M}^{\times n}$ are equivalent, making $\widehat{\mathcal{M}}$ into a special Γ -bicategory.

6. THE CONSTRUCTION OF $\tilde{\mathcal{N}}$

In this section we give the alternative version of the Γ -category for a permutative category \mathcal{N} enriched over simplicial categories.

We need a preliminary observation. Let $\underline{Hom}(X, Y)$ denote the internal hom in simplicial spaces. If X is (the nerve of) a simplicial category, then there is a unique composition of

paths:

$$\odot : \underline{Hom}(\Delta^1, X) \times_X \underline{Hom}(\Delta^1, X) \rightarrow \underline{Hom}(\Delta^1, X).$$

The uniqueness of the filling condition for all inner horns implies the uniqueness of filling for the spine, which in turn proves the following useful lemma.

Lemma 6.1. *Composition of paths is associative. In particular, there is a unique map*

$$\underline{Hom}(\Delta^1, X) \times_X \cdots \times_X \underline{Hom}(\Delta^1, X) \rightarrow \underline{Hom}(\Delta^1, X).$$

We will build a Γ -category $\tilde{\mathcal{N}}$ which will turn out to be equivalent to the standard Γ -category construction $\hat{\mathcal{N}}$. For the construction of $\tilde{\mathcal{N}}(\underline{n})$ we mimic that of bicategories in Section 5.

Construction 6.2. We build the Γ -category $\tilde{\mathcal{N}}$ as follows. The objects are given by $\{A_S, a_{S,T}\}$, with S and T as above; $A_S \in \text{Ob}\mathcal{N}$ and $a_{S,T} : A_{S \cup T} \rightarrow A_S \oplus A_T$ is an invertible morphism, that is, a 0-simplex in $\mathcal{N}(A_{S \cup T}, A_S \oplus A_T)$. We require conditions 1a-1d in the proof of Theorem 3.6 to hold. We note that the objects in $\tilde{\mathcal{N}}(\underline{n})$ are the same as the objects in the usual Γ -category construction $\hat{\mathcal{N}}(\underline{n})$.

Given two objects $\{A_S, a_{S,T}\}$ and $\{B_S, b_{S,T}\}$, the simplicial space of morphisms between them is defined as a subspace of

$$\prod_S \mathcal{N}(A_S, B_S) \times \prod_{S,T} \underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T}, B_S \oplus B_T)).$$

Let X be defined by the pullback

$$(6.3) \quad \begin{array}{ccc} X & \xrightarrow{\quad} & \prod_{S,T} \underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T}, B_S \oplus B_T)) \\ \downarrow & & \downarrow (d^1*, d^0*) \\ \prod_S \mathcal{N}(A_S, B_S) & \xrightarrow{\prod_{S,T} (a^*, b_*)} & \prod_{S,T} \mathcal{N}(A_{S \cup T}, B_S \oplus B_T)^{\times 2}, \end{array}$$

where the lower horizontal map corresponds to $(a_{S,T}^*, (b_{S,T})_*)$ for the S, T component.

Let Y be the equalizer

$$(6.4) \quad Y \longrightarrow X \xrightleftharpoons[p_2]{p_1} \prod_{S,T} \underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T}, B_S \oplus B_T)),$$

where p_1 is just the projection and p_2 is the projection composed with τ_* . This reproduces condition (5.4) in this setting.

We define $\tilde{\mathcal{N}}(\underline{n})(\{A_S, a_{S,T}\}, \{B_S, b_{S,T}\}) := Z$, where Z is the equalizer

$$(6.5) \quad Z \longrightarrow Y \xrightleftharpoons[q_2]{q_1} \prod_{S,T,U} \underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T \cup U}, B_S \oplus B_T \oplus B_U)),$$

where q_1 and q_2 are as defined below. This construction mimics the condition of equation (5.3).

Let f be the following composition

$$\begin{aligned}
& \mathcal{N}(A_U, B_U) \times \underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T}, B_S \oplus B_T)) \\
& \quad \downarrow \tilde{\oplus} \times \text{id} \\
& \underline{Hom}(\mathcal{N}(A_{S \cup T}, B_S \oplus B_T), \mathcal{N}(A_{S \cup T} \oplus A_U, B_S \oplus B_T \oplus B_U)) \\
& \quad \times \\
& \underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T}, B_S \oplus B_T)) \\
& \quad \downarrow \circ \\
& \underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T} \oplus A_U, B_S \oplus B_T \oplus B_U)) \\
& \quad \downarrow a^* \\
& \underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T \cup U}, B_S \oplus B_T \oplus B_U)),
\end{aligned}$$

where $\tilde{\oplus}, \circ, a$ denote the adjoint of \oplus , composition of maps, $a_{S \cup T, U}$, respectively.

We also have the map

$$\begin{aligned}
& \underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T \cup U}, B_{S \cup T} \oplus B_U)) \\
& \quad \downarrow b_* \\
& \underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T \cup U}, B_S \oplus B_T \oplus B_U)),
\end{aligned}$$

Note that the pullback conditions on X imply that $d^{0*}f$ is equal to $d^{1*}b_*$, thus, putting the two maps f and b_* together we get a map from Y to

$$\underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T \cup U}, B_S \oplus B_T \oplus B_U)) \times_{\mathcal{N}(A_{S \cup T \cup U}, B_S \oplus B_T \oplus B_U)} \underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T \cup U}, B_S \oplus B_T \oplus B_U)).$$

We can further compose with \odot , getting a map to $\underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T \cup U}, B_S \oplus B_T \oplus B_U))$.

Finally, we take the product over all (S, T, U) to get

$$q_1 : Y \longrightarrow \prod_{S, T, U} \underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T \cup U}, B_S \oplus B_T \oplus B_U)).$$

We define q_2 similarly, starting with

$$\mathcal{N}(A_S, B_S) \times \underline{Hom}(\Delta^1, \mathcal{N}(A_{T \cup U}, B_T \oplus B_U))$$

instead.

We now show that the collection of objects described above with the simplicial spaces of morphisms $\tilde{\mathcal{N}}(\underline{n})$ form a category enriched over simplicial spaces.

Given objects $A = \{A_S, a_{S, T}\}$, $B = \{B_S, b_{S, T}\}$, $C = \{C_S, c_{S, T}\}$, we define a composition map

$$\tilde{\mathcal{N}}(\underline{n})(B, C) \times \tilde{\mathcal{N}}(\underline{n})(A, B) \rightarrow \tilde{\mathcal{N}}(\underline{n})(A, C)$$

as follows: On one hand, we have a map given by the composition maps in \mathcal{N} :

$$\prod_S \mathcal{N}(A_S, B_S) \times \prod_S \mathcal{N}(B_S, C_S) \xrightarrow{\bullet} \prod_S \mathcal{N}(A_S, C_S).$$

Given (S, T) , let g be the following composition

$$\begin{aligned}
& \mathcal{N}(B_S, C_S) \times \mathcal{N}(B_T, C_T) \times \underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T}, B_S \oplus B_T)) \\
& \quad \downarrow \oplus \times \text{id} \\
& \mathcal{N}(B_S \oplus B_T, C_S \oplus C_T) \times \underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T}, B_S \oplus B_T)) \\
& \quad \downarrow \tilde{\bullet} \times \text{id} \\
& \underline{Hom}(\mathcal{N}(A_{S \cup T}, B_S \oplus B_T), \mathcal{N}(A_{S \cup T}, C_S \oplus C_T)) \\
& \quad \times \\
& \underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T}, B_S \oplus B_T)) \\
& \quad \downarrow \circ \\
& \underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T}, C_S \oplus C_T)).
\end{aligned}$$

Similarly, we let h be

$$\begin{aligned}
& \mathcal{N}(A_{S \cup T}, B_{S \cup T}) \times \underline{Hom}(\Delta^1, \mathcal{N}(B_{S \cup T}, C_S \oplus C_T)) \\
& \quad \downarrow \tilde{\bullet} \times \text{id} \\
& \underline{Hom}(\mathcal{N}(B_{S \cup T}, C_S \oplus C_T), \mathcal{N}(A_{S \cup T}, C_S \oplus C_T)) \\
& \quad \times \\
& \underline{Hom}(\Delta^1, \mathcal{N}(B_{S \cup T}, C_S \oplus C_T)) \\
& \quad \downarrow \circ \\
& \underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T}, C_S \oplus C_T)).
\end{aligned}$$

One can check that when we restrict the source of these maps to the pullback diagram (6.3), the condition $d^{0*}g = d^{1*}h$ is satisfied. Hence we can put g and h together to get a map to

$$\underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T}, C_S \oplus C_T)) \times_{\mathcal{N}(A_{S \cup T}, C_S \oplus C_T)} \underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T}, C_S \oplus C_T))$$

that we can later compose with \odot we get a map to

$$\underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T}, C_S \oplus C_T)).$$

Taking the product of these maps and the composition maps over S, T , we get a map

$$\tilde{\mathcal{N}}(\underline{n})(B, C) \times \tilde{\mathcal{N}}(\underline{n})(A, B) \xrightarrow{k} \prod_S \mathcal{N}(A_S, C_S) \times \prod_{S, T} \underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T}, C_S \oplus C_T)).$$

We want to show now that the image of k is contained in $\tilde{\mathcal{N}}(\underline{n})(A, C)$.

On the $\underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T}, C_S \oplus C_T))$ -component, $d^{1*}k = d^{1*}g$, which is equal to a^* on $\mathcal{N}(A_S, C_S) \times \mathcal{N}(A_T, C_T)$ by the pullback (6.3) on the space $\tilde{\mathcal{N}}(\underline{n})(A, B)$.

Similarly for $d^{0*}k$, we show it is the same as $c_{S, T*}$, thus showing we land in the pullback (6.3) for $\tilde{\mathcal{N}}(\underline{n})(A, C)$.

To show that conditions (6.4) and (6.5) hold we use the fact that they hold for $\tilde{\mathcal{N}}(\underline{n})(A, B)$ and $\tilde{\mathcal{N}}(\underline{n})(B, C)$ together with Lemma 6.1. We conclude that the composition is well defined.

The fact that composition in \mathcal{N} is associative and Lemma 6.1 imply that the composition is associative.

The identity of $\{A_S, a_{S,T}\}$ is the 0-simplex given by the collections of id_{A_S} together with the constant path at $a_{S,T} \in \mathcal{N}(A_{S \cup T}, A_S \oplus A_T)$.

We thus get a category $\tilde{\mathcal{N}}(\underline{n})$. We can extend this construction to a Γ -category in the usual way. Let $\theta : \underline{n} \rightarrow \underline{m}$ be a morphism in Γ . We construct the functor

$$\theta_* : \tilde{\mathcal{N}}(\underline{n}) \longrightarrow \tilde{\mathcal{N}}(\underline{m})$$

as follows.

We send the object $\{A_S, a_{S,T}\}$ of $\tilde{\mathcal{N}}(\underline{n})$ to the object $\{A_S^\theta, a_{S,T}^\theta\}$ of $\tilde{\mathcal{N}}(\underline{m})$, where $A_S^\theta = A_{\theta^{-1}S}$ and $a_{S,T}^\theta = a_{\theta^{-1}S, \theta^{-1}T}$.

For morphisms we construct a map

$$\tilde{\mathcal{N}}(\underline{n})(A, B) \longrightarrow \tilde{\mathcal{N}}(\underline{m})(A^\theta, B^\theta)$$

sending the projections of the $\theta^{-1}S$, $(\theta^{-1}S, \theta^{-1}T)$ -components to the S , (S, T) -components, respectively.

Consider the following diagram of functors

$$\begin{array}{ccc} \tilde{\mathcal{N}}(\underline{n}) & \xrightarrow{q} & \mathcal{N}^{\times n} \\ & \searrow i & \nearrow p \\ & \hat{\mathcal{N}}(\underline{n}) & \end{array}$$

j

Here q and p are the projections onto the $(\{1\}, \dots, \{n\})$ -components and i is the inclusion. It is clear that $qi = p$. On the other hand, j is the usual inverse equivalence for p ; we will describe j explicitly below.

Proposition 6.6. *The functors i and q are weak equivalences of categories (that is, they induce equivalences at the level of classifying spaces).*

Corollary 6.7. *The Γ -category $\tilde{\mathcal{N}}$ is special and it is levelwise equivalent to $\hat{\mathcal{N}}$, via the map i .*

Proof of Proposition 6.6. The proof will proceed as follows. Since i is the identity on objects and q is surjective on objects, it is enough to prove that both i and q induce weak equivalences for the simplicial spaces of morphisms.

We will first recall the definition of j in [SS, Lemma 2.2].

Given an object (A_1, \dots, A_n) in $\mathcal{N}^{\times n}$, we let

$$j(A_1, \dots, A_n) = \left\{ \bigoplus_{i \in S} A_i, e_{S,T} \right\}$$

where the sum is taken in the order of the indices of $S \subset \underline{n}$. The morphism $e_{S,T}$ is the uniquely determined isomorphism from $A_{S \cup T}$ to $A_S \oplus A_T$ given by composition of instances of τ .

For morphisms, we let

$$j(f_1, \dots, f_n) = \left\{ \bigoplus_{i \in S} f_i \right\}.$$

We then have that $pj = \text{id}$ and that there is a natural isomorphism $\lambda : \text{id} \rightarrow jp$ given on the object $\{A_S, a_{S,T}\}$ by the composition

$$A_S \rightarrow A_{\{i_1\}} \oplus A_{S-i_1} \rightarrow \cdots A_{\{i_1\}} \oplus \cdots \oplus A_{\{i_k\}}$$

of the corresponding a 's.

Given an object $A = \{A_S, a_{S,T}\}$ in $\hat{\mathcal{N}}(\underline{n})$ (and thus, in $\tilde{\mathcal{N}}(\underline{n})$), let $\dot{A} = \{\oplus_{i \in S} A_i, e_{S,T}^A\}$ denote its image under jq .

Given a pair (A, B) of objects in $\hat{\mathcal{N}}(\underline{n})$, consider the following diagram:

$$\begin{array}{ccc} \hat{\mathcal{N}}(\underline{n})(A, B) & \xrightarrow{i} & \tilde{\mathcal{N}}(\underline{n})(A, B) \\ \lambda \uparrow \cong & & \downarrow q \\ \hat{\mathcal{N}}(\underline{n})(\dot{A}, \dot{B}) & \xleftarrow[j]{\cong} & \prod_i \mathcal{N}(A_i, B_i) \end{array}$$

The map λ above is gotten by pre- and post-composition with the natural isomorphism λ already defined. We note that $\lambda jq i = \text{id}$ and $qi \lambda j = \text{id}$.

Thus, if we show that $i \lambda j q$ is homotopic to the identity we will have shown that $\lambda j q$ and $i \lambda j$ are homotopy inverses for i and q respectively, giving us the result we want.

To build a homotopy from $f = i \lambda j q$ to the identity, we show that there exists a map ϕ making the diagram below commute. For ease of notation we let $Z = \tilde{\mathcal{N}}(\underline{n})(A, B)$.

$$\begin{array}{ccc} & & Z \\ & \nearrow f & \uparrow d^{1*} \\ Z & \xrightarrow{\phi} & \underline{Hom}(\Delta^1, Z) \\ & \searrow \text{id} & \downarrow d^{0*} \\ & & Z \end{array}$$

More precisely, we will construct a map

$$\begin{array}{c} \prod_S \mathcal{N}(A_S, B_S) \times \prod_{S,T} \underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T}, B_S \oplus B_T)) \\ \downarrow \\ \prod_S \underline{Hom}(\Delta^1, \mathcal{N}(A_S, B_S)) \times \prod_{S,T} \underline{Hom}(\Delta^1 \times \Delta^1, \mathcal{N}(A_{S \cup T}, B_S \oplus B_T)), \end{array}$$

and show that when restricted to Z it lands on $\underline{Hom}(\Delta^1, Z)$.

We will first construct the map ϕ_S to the component $\underline{Hom}(\Delta^1, \mathcal{N}(A_S, B_S))$. Say $S = \{i_1 < \cdots < i_k\}$. Let $S_j = \{i_j, \dots, i_k\}$.

We consider the map g_j

$$\begin{array}{c}
\prod_{l=1}^{j-1} \mathcal{N}(A_{i_l}, B_{i_l}) \times \underline{Hom}(\Delta^1, \mathcal{N}(A_{S_j}, B_{i_j} \oplus B_{S_{j+1}})) \\
\downarrow \oplus \\
\mathcal{N}(\bigoplus_{l=1}^k A_{i_l}, \bigoplus_{l=1}^k B_{i_l}) \times \underline{Hom}(\Delta^1, \mathcal{N}(A_{S_j}, B_{i_j} \oplus B_{S_{j+1}})) \\
\downarrow \oplus \\
\underline{Hom}(\Delta^1, \mathcal{N}(A_{i_1} \oplus \cdots \oplus A_{i_{j-1}} \oplus A_{S_j}, B_{i_1} \oplus \cdots \oplus B_{i_j} \oplus B_{S_{j+1}})) \\
\downarrow (a^*)_*((b^{-1})_*)_* \\
\underline{Hom}(\Delta^1, \mathcal{N}(A_S, B_S)).
\end{array}$$

Here a is the composition of the instances of $a_{S,T}$ that take A_S to $A_{i_1} \oplus \cdots \oplus A_{i_{j-1}} \oplus A_{S_j}$, and similarly for b taking B_S into $B_{i_1} \oplus \cdots \oplus B_{i_j} \oplus B_{S_{j+1}}$.

When we restrict to Z , $d^1 g_j = d^0 g_{j+1}$ and thus we get a map into

$$\underline{Hom}(Spine_{|S|}, \mathcal{N}(A_S, B_S)).$$

By Lemma 6.1, this extends to $\underline{Hom}(\Delta^1, \mathcal{N}(A_S, B_S))$. The result is the map ϕ_S .

From the construction it is clear that $d^1 \phi_S = f$ and $d^0 \phi_S = \text{id}$ on this component.

Now, we look at the component $\underline{Hom}(\Delta^1 \times \Delta^1, \mathcal{N}(A_{S \cup T}, B_S \oplus B_T))$.

Let g be the composition

$$\begin{array}{c}
Z \\
\downarrow \phi_S \times \phi_T \\
\underline{Hom}(\Delta^1, \mathcal{N}(A_S, B_S)) \times \underline{Hom}(\Delta^1, \mathcal{N}(A_T, B_T)) \\
\downarrow \oplus * \\
\underline{Hom}(\Delta^1, \mathcal{N}(A_S \oplus A_T, B_S \oplus B_T)) \\
\downarrow (a^*)_* \\
\underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T}, B_S \oplus B_T)).
\end{array}$$

Note that g and the projection $Z \rightarrow \underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T}, B_S \oplus B_T))$ give composable paths. By Lemma 6.1 and the given conditions on Z , the composed path is equal to

the map h :

$$\begin{array}{c}
Z \\
\downarrow \phi_{S \cup T} \\
\underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T}, B_{S \cup T})) \\
\downarrow b_* \\
\underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T}, B_S \oplus B_T)).
\end{array}$$

We can put these maps together and think of them as giving a map into

$$\underline{Hom}(\Delta^1 \times \Delta^1, \mathcal{N}(A_{S \cup T}, B_S \oplus B_T)),$$

since the relevant paths are equal:

$$\begin{array}{ccc}
\bullet & \xrightarrow{\text{proj}} & \bullet \\
g \downarrow & & \downarrow h \\
\bullet & \xrightarrow{id} & \bullet
\end{array}$$

Using the conditions on Z , it is easy to check that this construction yields the desired map

$$Z \rightarrow \underline{Hom}(\Delta^1, Z)$$

which restricts to the identity and f at each end and gives the homotopy we wanted. \square

7. PROOF OF THEOREM 3.7

In this section we prove Theorem 3.7 which relates the Γ -bicategory construction of Theorem 3.6 and the construction of $\tilde{\mathcal{N}}$ from the previous section.

Proof of Theorem 3.7. We will build a map of bisimplicial spaces from the levelwise nerve of the simplicial category $S\widehat{\mathcal{M}}(\underline{n})$ to $N\tilde{\mathcal{N}}(\underline{n})$. Since $N\mathcal{M}(A, B)$ maps into $\mathcal{N}(A, B)$ we can think of 1-morphisms in \mathcal{M} as morphisms in \mathcal{N} . Thus we can think of objects in $\widehat{\mathcal{M}}(\underline{n})$ as objects in $\tilde{\mathcal{N}}(\underline{n})$.

The $(0, -)$ simplicial spaces for $NS\widehat{\mathcal{M}}(\underline{n})$ and $N\tilde{\mathcal{N}}(\underline{n})$ are given by the objects of $\widehat{\mathcal{M}}(\underline{n})$ and $\tilde{\mathcal{N}}(\underline{n})$, respectively. The desired map at the $(0, -)$ level is obtained by the identification of objects of $\widehat{\mathcal{M}}(\underline{n})$ as objects of $\tilde{\mathcal{N}}(\underline{n})$.

Recall that a 1-morphism in $\widehat{\mathcal{M}}(\underline{n})$ is given by a collection $\{f_S, \phi_{S,T}\}$ of 1-morphisms and 2-morphisms in $\mathcal{M}(\mathcal{R})$. By considering the maps $N\mathcal{M}(A, B) \rightarrow \mathcal{N}(A, B)$, we can think of $\phi_{S,T}$ as a 0-simplex in $\underline{Hom}(\Delta^1, \mathcal{N}(A_{S \cup T}, B_S \oplus B_T))$. As noted throughout the construction, the restrictions on $\tilde{\mathcal{N}}(\underline{n})(A, B)$ reflect the coherence axioms for $\{f_S, \phi_{S,T}\}$, so in general, we can think of a 1-morphism in $\widehat{\mathcal{M}}(\underline{n})$ as a 0-simplex in $\tilde{\mathcal{N}}(\underline{n})(A, B)$. Similarly, we can think of a 2-morphism in $\widehat{\mathcal{M}}(\underline{n})$ as a 1-simplex.

We can thus construct a map

$$NS\widehat{\mathcal{M}}(\underline{n})_{p,q} \rightarrow N\tilde{\mathcal{N}}(\underline{n})_{p,q}$$

as follows.

Recall that a (p, q) -simplex in $NS\widehat{\mathcal{M}}(\underline{n})$ is given by a collection $\{A_i\}_{i=0}^p$ of objects in $\widehat{\mathcal{M}}(\underline{n})$, diagrams of the form

$$\begin{array}{ccc} & A_j & \\ f_{ij}^l \nearrow & & \searrow f_{jk}^l \\ A_i & \xrightarrow{f_{ik}^l} & A_k, \\ & \Downarrow \varphi_{ijk}^l & \end{array} \quad \text{for all } 0 \leq i < j < k \leq p, 0 \leq l \leq q$$

subject to the coherence conditions in (A.3), together with coherent 2-morphisms $f_{ij}^l \Rightarrow f_{ij}^{l+1}$.

We can map this to

$$\coprod_{A_0, \dots, A_p} \tilde{\mathcal{N}}(\underline{n})(A_0, A_1) \times \dots \times \tilde{\mathcal{N}}(\underline{n})(A_{p-1}, A_p)$$

by projecting the $(i, i+1)$ -entries and using the identification of 1-morphisms and 2-morphisms in $\widehat{\mathcal{M}}(\underline{n})$ as 0-simplices and 1-simplices of $\tilde{\mathcal{N}}(\underline{n})$ described above.

It is clear that this map extends to a map of Γ -spaces. \square

APPENDIX A. CLASSIFYING SPACES OF BICATEGORIES

Categories are closely related to spaces through the classifying space construction. To every category we can assign a space. This assignment gives a functor that is part of a Quillen equivalence between Thomason's model structure on the category of small categories [Tho] and the usual model structure in the category of topological spaces.

The same can be done with bicategories. In fact, there are many distinct constructions of the classifying space of a bicategory ([CCG]). All these constructions give equivalent spaces, in the non-enriched case. Here we describe the version of the nerve that was used in the preceding sections.

Lack and Paoli [LP] introduce a version of a nerve of non-enriched bicategories that gives rise to a simplicial object in the category of small (non-enriched) categories. The construction can be extended to the enriched case, giving a simplicial object in Cat , where Cat is the category of small categories enriched over simplicial sets. This construction is closely related to the bar construction for monoidal categories defined in [BDR], as we pointed out. This nerve is called *2-nerve* in [LP] and *Segal nerve* in [CCG].

Definition A.1. Let \mathcal{C} be a bicategory. The *Segal nerve* \mathbf{NC} is the simplicial object in Cat given by normal homomorphisms, that is,

$$\mathbf{N}_n \mathcal{C} = \underline{NorHom}([n], \mathcal{C}),$$

where the objects are normal pseudofunctors and the morphisms are *icons*.

Normal pseudofunctors are those for which the identity natural isomorphism is the identity. An *icon* (Identity Component Oplax Natural transformation) is an oplax natural transformation (see [Lei, Section 1.2]) such that the map $\eta_A : \mathcal{F}A \rightarrow \mathcal{G}A$ is the identity, so in particular, we require that $\mathcal{F}A = \mathcal{G}A$. We now unravel the definition above.

An object of $\mathbf{N}_n\mathcal{C}$ is given then by a collection of diagrams

$$(A.2) \quad \begin{array}{ccc} & C_j & \\ f_{ij} \nearrow & & \searrow f_{jk} \\ C_i & \xrightarrow{f_{ik}} & C_k, \\ & \Downarrow \varphi_{ijk} & \end{array} \quad \text{for all } 0 \leq i < j < k \leq n,$$

where φ_{ijk} is an invertible 2-morphism. This collection must satisfy the following coherence condition for all $0 \leq i < j < k < l \leq n$:

$$(A.3) \quad \begin{array}{ccc} & C_j & \\ & \Downarrow \varphi_{ijk} & \\ C_i & \xrightarrow{\varphi_{ikl}} & C_k \\ & \searrow & \nearrow \\ & C_l & \end{array} = \begin{array}{ccc} & C_j & \\ & \Downarrow \varphi_{ijl} & \\ C_i & \xrightarrow{\varphi_{jkl}} & C_k \\ & \searrow & \nearrow \\ & C_l & \end{array}.$$

Given objects $\{C_i, f_{ij}, \varphi_{ijk}\}$ and $\{C_i, f'_{ij}, \varphi'_{ijk}\}$ (note that the collections of objects are equal), a morphism between them is given by a collection of 2-morphisms $\eta_{ij} : f_{ij} \Rightarrow f'_{ij}$ for $i \leq j$, such that some coherence conditions ([CCG, Eq. (44)]) are satisfied.

Remark A.4. We note that the bar construction for monoidal categories of [BDR] is equal to the Segal nerve. More precisely, if \mathcal{M} is a monoidal category, then the simplicial category BM of [BDR] is equal to $\mathbf{N}\Sigma\mathcal{M}$ (with a possible reordering of the indices).

The Segal nerve is functorial with respect to normal pseudofunctors. It is the case that any pseudofunctor can be normalized ([LP, Prop. 5.2]).

It is clear from the definition that the Segal nerve preserves products.

Definition A.5. Let \mathcal{C} be a bicategory. The *classifying space* of \mathcal{C} is the realization $|\mathbf{NC}|$.

Let $\mathcal{F}, \mathcal{G} : \mathcal{C} \rightarrow \mathcal{D}$ be pseudofunctors, and $\eta : \mathcal{F} \rightarrow \mathcal{G}$ a transformation. As pointed out in the proof of [CCG, Prop. 7.1], these data gives rise to a pseudofunctor

$$\mathcal{H} : \mathcal{C} \times \mathbf{1} \rightarrow \mathcal{D}$$

that restricts to \mathcal{F} and \mathcal{G} at 0 and 1. This pseudofunctor can be normalized, yielding the following result:

Proposition A.6. A transformation between pseudofunctors $\mathcal{F}, \mathcal{G} : \mathcal{C} \rightarrow \mathcal{D}$ gives rise to a homotopy between the maps

$$|\mathbf{NF}|, |\mathbf{NG}| : |\mathbf{NC}| \rightarrow |\mathbf{ND}|.$$

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